Army Materiel Systems Analysis Activity

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ON SOME PROPERTIES OF A CLASS OF RELIABILITY GROWTH PLANNING MODELS

AUGUST 2013

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14. ABSTRACT
This report addresses four properties of PM2-Continuous that were identified in a review as being counterintuitive. It will be shown that these properties are shared by reliability growth planning models that are based on a large class of B-mode rate of occurrence functions h(t) that were advocated by Professor Douglas Miller for reliability growth modeling. The report examines the model parameter trade-offs and their effects on the properties for scenarios considered in the review and for scenarios that are germane to conducting a reliability growth test.

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LIST OF ACRONYMS

ACAT 1	- Acquisition Category 1
AMSAA	- US Army Materiel Systems Analysis Activity
ASA(ALT)	- Assistant Secretary of the Army for Acquisition, Logistics , and Technology
ATEC	- Army Test and Evaluation Command
CAP	- Corrective Action Period
COTS	- Commercial-off-the-Shelf
DFR	- Design for Reliability
DT	- Developmental Test
EMD	- Engineering, Manufacturing, Development
EOS	- Exponential Order Statistic
FEF	- Fix Effectiveness Factor
IOT	- Initial Operational Test
LAD	- Logistics Analysis Division
DTM	- Directive-Type Memorandum
DoD	- Department of Defense
M_G	- MTBF Goal (in DT)
$M_{G,IOT}$	- MTBF Goal (in IOT)
M_R	- MTBF Requirement
M_I	- MTBF Initial
MIL-HDBK	- Military Handbook
MS	- Management Strategy
MTBF	- Mean Time Between Failures
NHPP	- Nonhomogeneous Poisson Process
PMs	- Program Managers
PM2	- Planning Model based on Projection Methodologies
RG	- Reliability Growth
RGC	- Reliability Growth Curve
SEP	- Systems Engineering Plan
TEMP	- Test and Evaluation Master Plan
TR	- Technical Report

PROPERTIES OF A CLASS OF RELIABILITY GROWTH PLANNING MODELS

1. EXECUTIVE SUMMARY

DoD Directive-Type Memorandum (DTM) 11-003- Reliability Analysis, Planning, Tracking, and Reporting, March 21, 2011 [1], applies to all major DoD developmental acquisition programs. This DTM requires that reliability growth curves (RGC) for such programs be included in the Systems Engineering Plan (SEP) at Milestone A, and be updated in the Test and Evaluation Master Plan (TEMP) beginning at Milestone B. The RGC is to "reflect the reliability growth strategy and be employed to plan, illustrate, and report reliability growth." Additionally, the Office of the Assistant Secretary of the Army for Acquisition, Logistics , and Technology (ASA(ALT)) issued a Memorandum dated June 26, 2011 [2], addressing the subject "Improving the Reliability of U.S. Army Materiel Systems." This document states that "Program Managers (PMs) of all Acquisition Category I (ACAT I) systems and for ACAT II systems where the sponsor has determined reliability to be an attribute of operational importance shall place reliability growth planning curves in the SEP, TEMP, and Engineering and Manufacturing (EMD) contracts and ensure that U.S. Army systems are resourced to accomplish this requirement." The ASA(ALT) document stipulates that "Reliability growth planning is quantified and reflected through a reliability growth planning curve using the Planning Model based on Projection Methodology (PM2)." The document also states "Where warranted by unique system characteristics, the Army Test and Evaluation Command (ATEC), in consultation with the Project Manager, may specify an alternative reliability growth planning method."

Due to the focus on the PM2 reliability growth planning model [3,4] in the ASA(ALT) document, it has become widely used. Recently, an Army organization that wished to utilize an alternate method had a consultant conduct a review of this planning model. A number of properties of PM2 were identified by the organization and their consultant that were deemed to be in conflict with their engineering experience. These model properties were examined by the consultant for the continuous version and discrete version of PM2. The continuous version addresses the case where growth test duration is measured on a continuous scale such as simulated mission hours or miles. The discrete version applies to the case where test duration is measured in a discrete scale such as trials or rounds. The consultant concluded that four of the five properties examined for the continuous version "are counterintuitive and are not properties typically desired to be in a reliability growth planning model." For the discrete version, four properties were examined that closely align with four corresponding continuous model properties examined. The consultant again concluded that three of these discrete properties "are counterintuitive and are not properties typically desired to be in a reliability growth planning model." Since the PM2 discrete version properties examined mimic the corresponding continuous version properties, only the continuous version of PM2 is addressed in this paper.

These properties for the PM2-Continuous Model will be examined in this report for a large class of problem failure mode expected rate of occurrence functions, which includes the PM2 function. As noted in Section 3, this large class of functions includes the problem mode expected rate of occurrence functions for a family of growth models advocated by the late Professor Miller [5] for use in modeling the occurrence of new software bugs in reliability growth testing. It will be demonstrated that the "counterintuitive" properties noted in the review

of PM2 are shared by reliability growth planning models that are based on this large class of problem failure mode rate of occurrence functions. Furthermore, it will be shown that the review's label of "counterintuitive" for the first three examined properties is based on sensitivity analyses that induce tradeoffs between growth parameters. It will be demonstrated that the induced tradeoffs do not conform to the conditions under which a reliability growth test is conducted. The last "counterintuitive" property examined is shown to follow from general guidance given in a PM2 software program regarding a planning risk. Adhering to the guidance yields a constraint associated with the initial failure rate. However, following the guidance would lead to exactly the same constraint for a large class of growth models that includes PM2. This class of planning models is even broader than the class mentioned above and further considered in Sections 3 through 6 with regard to discussing the first three "counterintuitive" properties. In fact, as shown in Section 7, the resulting associated "counterintuitive" property of PM2 that occurs if one decides to adhere to this risk criterion is shared by every reliability growth planning model that utilizes a function that is a suitable candidate for portraying the rate of occurrence of problem failure modes.

2. INTRODUCTION

Reliability Growth (RG) planning is an area of reliability growth that addresses program schedules, amount of testing, resources available, and the realism of the test program in achieving its goals. Reliability growth planning is quantified and reflected through a reliability growth program planning curve. Planning curves typically express a measure of reliability such as Mean Time Between Failure (MTBF) as a function of cumulative test duration and other programmatic resources. The measure of test duration for MTBF is test simulated mission hours. A RG planning curve serves as a baseline against which reliability assessments may be compared throughout the test program. The key features of a planning curve include the following: (1) The initial MTBF, M_I, and the goal MTBF, M_G; (2) Test phases with MTBF steps over each test phase, where the step represents the benchmark MTBF for the test phase; (3) Corrective Action Periods (CAPs) which are scheduled calendar periods between test phases for implementing corrective actions (termed fixes). Most growth often occurs due to corrective actions in the CAPs, although some fixes may occur in the test phases; (4) An idealized MTBF curve from which the benchmark MTBF steps over the test phases are obtained. A sample reliability growth curve and its features are illustrated in Figure 1.

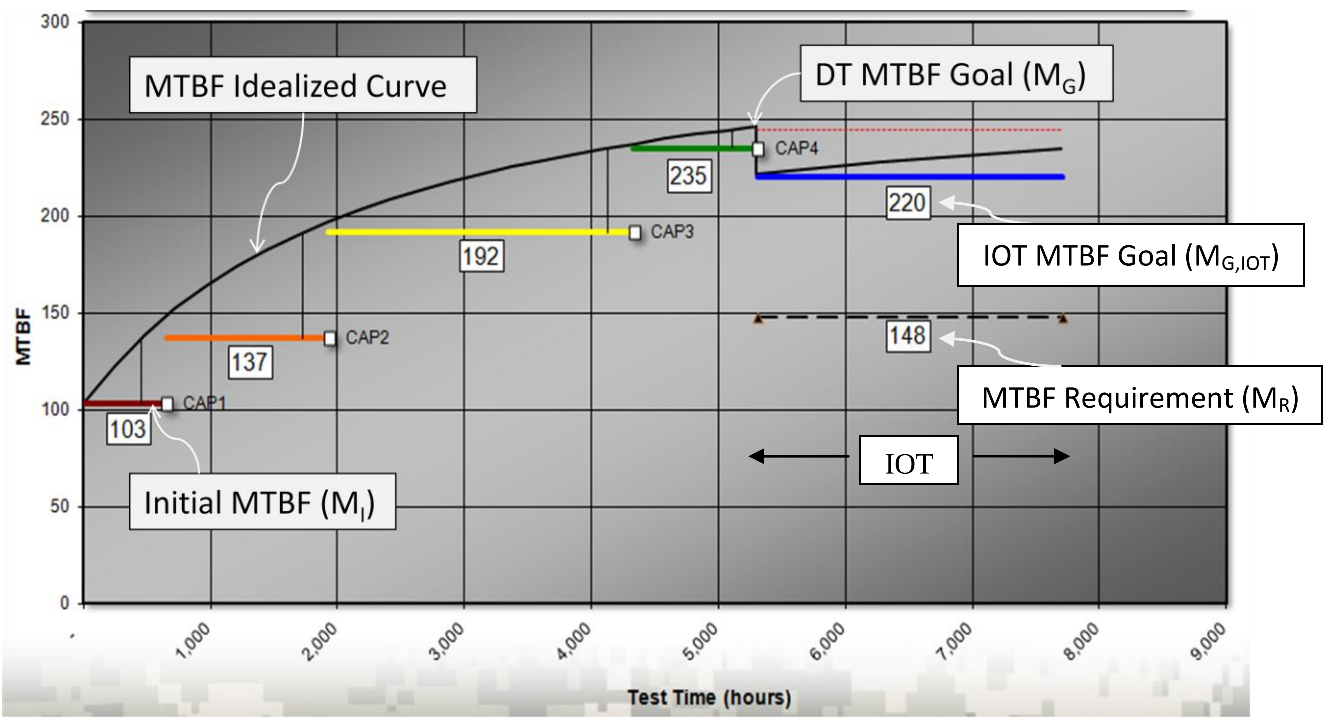

Figure 1. Reliability Growth Planning Curve

The idealized MTBF curve at cumulative test time t represents the MTBF that can be obtained if all the problem failure modes discovered in test by time t have corrective actions implemented by t. The steps are drawn underneath the idealized curve to reflect an average lag time from when a problem failure mode is first observed to when a corrective action can be physically implemented. More details can be found in [3,4]. To obtain achievable MTBF steps, the idealized MTBF curve should be expressed in terms of achievable input planning parameters that are under the control of the Program Office. PM2 utilizes such planning parameters.

The key planning parameters in the PM2 model are: (1) The initial MTBF goal to be obtained by scheduled and funded design for reliability activities prior to the growth test developmental period; (2) The goal developmental test (DT) MTBF that supports achieving a successful demonstration of the MTBF requirement with a measure of assurance (e.g. statistical confidence); (3) The developmental test time, T, over which problem failure modes are discovered that are expected to be addressed and implemented prior to or during the last DT CAP; (4) The planned management strategy, MS, defined to be the expected fraction of the initial failure intensity due to failure modes that will be addressed by corrective actions if seen during the DT. Such modes are termed B-modes [3] and are the modes that were previously referred to as problem modes. The remaining failure modes are termed A-modes [3]; and (5) The average fix effectiveness factor expected to be achieved during the DT program. For an individual B-mode, the fraction reduction in the initial rate of occurrence of failures due to the mode as a result of a fix is called the mode's fix effectiveness factor (FEF). The expected average of these mode FEFs is the planning parameter referred to in (5) above.

Section 3 provides several key reliability growth planning concepts associated with the idealized MTBF growth curve. Also the assumed properties of the B-mode rate of occurrence function, denoted by h(t), are stated and motivated. In Sections 4 - 7 it is shown how replacing the h(t) utilized in PM2 by any h(t) in the class of rate of occurrence functions that satisfy the assumed properties, which includes the PM2 h(t), gives rise to a reliability growth model that has the same four properties deemed counterintuitive in the review and gives rise to the three areas of concern noted by the Army organization. Section 8 contains remarks about the addressed properties based on the facts demonstrated in the report.

3. BACKGROUND

The MTBF idealized curve mentioned above is generated by considering the two types of failure modes defined in Section 2, i.e., the A-modes and B-modes. The MTBF idealized curve is generated through the following equation for its reciprocal, the rate of occurrence of all failures (termed the failure intensity)

$$\rho(t) = \lambda_A + (1 - \mu_d)\{\lambda_B - h(t)\} + h(t) \tag{1}$$

In the above equation, h(t) is the rate of occurrence of B-modes at time t. It is also equal to the expected failure intensity due to the B-modes not surfaced by time t. Using this interpretation of h(t), the last term is the contribution to the failure intensity due to these unobserved B-modes. In this equation λ_B equals h(0), the expected failure intensity due to all the B-modes at the start of the first test phase. Thus $\lambda_B - h(t)$ represents the expected failure intensity due to the B-modes that were surfaced by time t. In the second term, μ_d denotes the average planned fix effectiveness factor. Thus, the second term represents the failure intensity contribution due to the B-modes surfaced by time t after they have been corrected with the planned fix effectiveness factor. Finally, the first term represents the failure intensity contribution due to the A-modes.

An important concept associated with reliability growth planning is the MTBF growth potential and its reciprocal, the failure intensity growth potential. These are denoted by M_{GP} and ρ_{GP}, respectively. The value ρ_{GP} is approached in the limit as h(t) goes to zero. Thus, M_{GP} can be viewed as a ceiling on the MTBF that can be achieved for a given design. By Equation (1), ρ_{GP} and its reciprocal M_{GP} are given as follows:

$$\rho_{GP} = \lambda_A + (1 - \mu_d)\lambda_B \quad , \quad M_{GP} = \frac{M_I}{1 - (MS)\mu_d} \tag{2}$$

These equations and concepts not only apply to the PM2 planning model, but are equally valid for any reliability growth planning model that utilizes an appropriate form for h(t). For PM2, h(t) is given by

$$h(t) = \frac{\lambda_B}{1 + \beta t} \tag{3}$$

where β is a positive scale parameter for h(t).

In Sections 4 - 7, the properties of PM2 that were deemed counterintuitive will be discussed for a class of growth planning models that are generated via Equation (1) by a large set of h(t) functions. It will be assumed that the h(t) in this class of functions satisfies a set of assumptions that include the fundamental properties advocated by Miller [5]. Miller deemed these properties appropriate for modeling the pattern of the rate of occurrence of new software bugs for reliability growth models. These fundamental properties can be motivated by first considering the exact expression for the expected failure intensity due to the B-modes not surfaced by t, denoted $h_{ex}(t)$. To do so, let k be the number of B-modes residing in the system at

the start of the test period, and let λ_i be the rate of occurrence of failures due to B-mode i at the start of the test period prior to corrective actions. It can be shown [6]

$$h_{ex}(t) = \sum_{i=1}^{k} \lambda_i e^{-\lambda_i t} \tag{4}$$

It is assumed that at least one of the λ_i is positive.

Note the following fundamental properties of $h_{ex}(t)$: (i) $h_{ex}(0)$ is a finite positive number; (ii) $h_{ex}(t)$ approaches 0 as t increases; and (iii) $(-1)^n h_{ex}^{(n)}(t) > 0$ for all nonnegative integers n, where $h_{ex}^{(n)}(t)$ denotes $h_{ex}(t)$ for n = 0 and the n'th derivative with respect to t of $h_{ex}(t)$ for positive n. A function that has property (iii) is called completely monotone [5]. Observe $h_{ex}(t)$ has too many parameters for use in a reliability growth planning model. Thus, for complex systems, a parsimonious approximation to $h_{ex}(t)$ is typically utilized. In the remainder of the paper h(t) will denote a parsimonious approximation for $h_{ex}(t)$. It will be assumed that h(t) satisfies the fundamental properties (i) through (iii) above. It will also be assumed that (iv) the parameters for h(t) include a scale parameter, $\beta > 0$, and expected initial B-mode rate of occurrence, $\lambda_B = h(0)$. Note, by definition, β is a scale parameter of h(t) if and only if there exists a function g for which $g(\beta t) = h(t)$ for all positive β and nonnegative t for each minimal fixed set of parameters that, in addition to scale parameter β, define h. As implied above, the minimal parameter set need not be limited to just h(0) and the scale parameter. However, it is assumed the minimal set of parameters that define h includes β and λ_B, and can be specified independently within their defined parameter domains, in the absence of any imposed conditions other than satisfying (i) through (iii). It is interesting to observe that the power law function, which has been used for reliability projection [3], $h(t) = \lambda\omega\, t^{\omega-1}$, where $\lambda > 0$ and $0 < \omega < 1$, can be reparameterized in terms of a scale parameter, $s > 0$, and ω. In this parameterization, $h(t) = \omega(st)^{\omega-1}$ where $s = \lambda^{-\{1/(1-\omega)\}}$. Observe there is a one-to-one correspondence between $(s,\omega) \epsilon\, \Theta$ and $(\lambda,\omega) \epsilon\, \Theta$ where $\Theta = \{(x,y)|0 < x \text{ and } 0 < y < 1\}$. Note however, the power law h(t) does not satisfy Property (i). Also note the PM2 h(t) function given in Equation (3) is parameterized in terms of the scale parameter $\beta > 0$ and $\lambda_B = h(0)$. To specify the next assumption for h(t), let $\mu(t)$ denote the expected number of B-modes surfaced by test time t. Thus $\mu(t) = \int_0^t h(x)dx$. Observe $h(t) \div \{\mu(t)/t\} < 1$ for all $t > 0$. The assumption is that (v) $\lim_{t\to\infty}[h(t) \div \{\mu(t)/t\}] < 1$, where the parameters that define h are held constant as $t\to\infty$. By Proposition A.1 in Appendix A it follows that this limit equals 0 when $\lim_{t\to\infty} \mu(t) < \infty$. Also, when $\lim_{t\to\infty} \mu(t) = \infty$ and $t\{h(t)\}$ is bounded above by a finite value one has $\lim_{t\to\infty}[h(t) \div \{\mu(t)/t\}] = 0$. For the h(t) utilized by PM2 the latter situation occurs and hence the limit of the ratio $h(t)\div\{\mu(t)/t\}$ goes to 0 as $t\to\infty$. For the power law $h(t) = \lambda\omega\, t^{\omega-1}$, one has $\lim_{t\to\infty}[h(t) \div \{\mu(t)/t\}] = \omega$. Note to have the rate of occurrence of B-modes approach zero as $t \to \infty$ requires $0 < \omega < 1$. This choice of h(t) with $\omega < 1$ satisfies the above limit assumption, even though h(0) is not finite. The final assumption is that (vi) h(t) can be expressed as the product $\lambda_B\{h_0(t)\}$, where $h_0(t)$ does not depend on λ_B. The PM2 h(t) satisfies this as well. More generally, the h(t) function for each member of the family of Gamma Exponential Order Statistic (EOS) Models considered by Miller [5,Section 5] satisfy Assumption (vi) as well as Assumptions (i) through (v) above. The h(t) function for this family can be expressed as follows:

$$h(t) = \frac{\lambda_B}{(1+\beta t)^{\alpha+2}} \tag{5}$$

where $\alpha > -2$. Note β in Equation (5) equals the reciprocal of the β parameter in [5] and the α in (5) is one less than the α parameter in [5].

For $\alpha > -1$, the limit of the expected number of B-modes as $t \to \infty$ is finite. In this case the model is referred to as a Pareto Nonhomogeneous Poisson Process (NHPP) [5, Section 5]. When $\alpha > -1$, if $k = \lambda_B \div \beta(\alpha+1)$ is a positive integer, then h(t) can also represent the expected rate of occurrence function for B-modes associated with an Independent and Identically Distributed Order Statistic (IIDOS) process [5] .For this IIDOS process the B-mode initial failure rates are viewed as the realization of a random sample of size k drawn from a gamma distribution with mean equal to $\beta(\alpha+1)$ and variance $\beta^2(\alpha+1)$. The h(t) function for this case corresponds to the h(t) of the finite-k AMSAA Maturity Projection Model (AMPM) in [6]. If one re-parameterizes h(t) in (5) in terms of λ_B , β, and k and utilizes the domains $\lambda_B > 0$, $\beta > 0$ and k a positive integer then, with respect to these parameters and associated domains, h(t) satisfies assumptions (i) through (vi). The PM2 h(t) function corresponds to the case where $\alpha = -1$. For this case, the limit of the expected number of B-modes as $t \to \infty$ is infinite. This form for h(t) was utilized by Musa and Okumoto [7] as the rate of occurrence of software failures in a NHPP model. For $-2 < \alpha < -1$ the expected number of B-modes is also infinite. The resulting model is termed a Generalized Power Law NHPP Model [5]. In [5, Section 5], the power law $\mu(t)$ function is shown to be a limit of $\mu(t)$ functions that belong to Generalized Power Law NHPP Models. From (5), one can show $\mu(t)$ for a Generalized Power Law NHPP Model is as follows:

$$\mu(t) = \left\{\frac{\lambda_B}{-(\alpha+1)\beta}\right\}\left\{(1+\beta t)^{-(\alpha+1)} - 1\right\} \tag{6}$$

To obtain the power law expected B-mode function, the limit of $\mu(t)$ in (6) is taken as β approaches ∞. To avoid degenerate limits that correspond to either $\mu(t) = \infty$ or $\mu(t) = 0$ for all $t \geq 0$, the limit is taken subject to $\mu(t_0) = m_0$ where $t_0 > 0$ and m_0 is a positive integer. These parameters are held fixed, along with $\alpha \in (-2,-1)$, as $\beta \to \infty$. From the condition that $m_0 = \mu(t_0)$ and (6) one obtains

$$\lambda_B = \frac{-(\alpha+1)\beta m_0}{\{(1+\beta t_0)^{-(\alpha+1))} - 1\}} \tag{7}$$

Substituting the expression for λ_B in (7) into (6) yields

$$\mu(t) = m_0\left[\frac{(1+\beta t)^{-(\alpha+1)} - 1}{(1+\beta t_0)^{-(\alpha+1)} - 1}\right] \tag{8}$$

To emphasize that t_0, m_0 and α in (8) are held fixed as $\beta \to \infty$, the $\mu(t)$ in (8) will be denoted by $\mu(t; t_0, m_0, \alpha, \beta)$. Note

$$\mu(t; t_0, m_0, \alpha, \beta) = m_0 \left[\left\{ \frac{1+\beta t}{1+\beta t_0} \right\}^{-(\alpha+1)} \left\{ \frac{1 - \frac{1}{(1+\beta t)^{-(\alpha+1)}}}{1 - \frac{1}{(1+\beta t_0)^{-(\alpha+1)}}} \right\} \right] \tag{9}$$

Also observe

$$-2 < \alpha < -1 \leftrightarrow -1 < \alpha + 1 < 0 \leftrightarrow 0 < -(\alpha + 1) < 1 \tag{10}$$

It follows from (9) and (10) that

$$\mu(t; t_0, m_0, \alpha) = \lim_{\beta \to \infty} \mu(t; t_0, m_0, \alpha, \beta) = m_0 \left\{ \left(\frac{t}{t_0} \right)^{-(\alpha+1)} \right\} = \lambda t^{\omega} \tag{11}$$

where

$$\lambda = \frac{m_0}{t_0^{-(\alpha+1)}} \;\; and \;\; \omega = -(\alpha + 1) \epsilon \, (0, 1) \tag{12}$$

Setting $t_0 = 1$, from (12) one obtains

$$\lambda = m_0 \;\; and \;\; \omega = -(\alpha + 1) \tag{13}$$

Equations (11), (12) and (13) show that each power law mean value function $\mu(t) = \lambda t^{\omega}$ where $\lambda > 0$ and $0 < \omega < 1$ can be obtained as the limit of Generalized Power Law NHPP Models that satisfy Assumptions (i) through (vi) above. Note the associated power law expected B-mode rate of occurrence function is $h(t) = \lambda\omega t^{\omega-1} = m_0\{-(\alpha + 1)\} t^{-(\alpha+1)-1}$ for $t > 0$. Observe the power law itself does not correspond to a member of the Generalized Power Law NHPP family since h(t) is not finite at the origin. It is interesting to note that the Generalized Power Law Models provide examples of h(t) functions that, unlike the power law h(t), satisfy all the assumptions (i) through (vi) above and, for Assumption (v) , $\lim_{t\to\infty}[\,h(t) \div \{\,\mu(t)/t\}] = -(\alpha + 1)$ where $0 < -(\alpha + 1) < 1$. Miller conjectures [5, pg.59] that the family of Gamma Exponential Order Statistic Models "may be rich enough to represent adequately the entire family of EOS Models (defined in [5])." With respect to the family of EOS Models, Miller [5, pg.59] states "The possible patterns of underlying failure rates are unlimited."

To obtain the associated reliability growth model, the parsimonious approximation, h(t), for the expected B-mode rate of occurrence function is utilized in Equation (1). The resulting idealized failure intensity would be used with test phases, specified lag times, and the program test schedule to obtain the test phase MTBF benchmarks and associated metrics as described in MIL-HDBK 189-C [3] for the PM2 h(t) function. As indicated above, the PM2 h(t) is one member of the large class of h(t) functions that satisfy Assumptions (i) through (vi) above.

4. PROPERTY 1 – EFFECT OF VARYING TEST TIME FOR FIXED GOAL MTBF

The first property, corresponding to the first topic of concern of the Army organization sponsor of the PM2 review, can be paraphrased as follows: Every growth curve will achieve the DT goal MTBF at time T, for any T. This statement is correct from the point of view that as long as the DT goal MTBF is less than the growth potential MTBF, expressed in Equation (2) in terms of input parameters, then one can construct a growth curve that goes from the user input initial MTBF to the goal MTBF in the specified time T with user prescribed values for the management strategy and average FEF. This was one of the model properties deemed counterintuitive by the consultant and will be referred to as Property 1 in the paper. This scenario in which T varied and the other planning parameters were held fixed, forces a tradeoff between T and the model scale parameter β, as illustrated in Figure 2. Here, β denotes the scale parameter for h(t), where h(t) is assumed to meet the assumptions of the class of B-mode rate of occurrence functions described in Section 3.

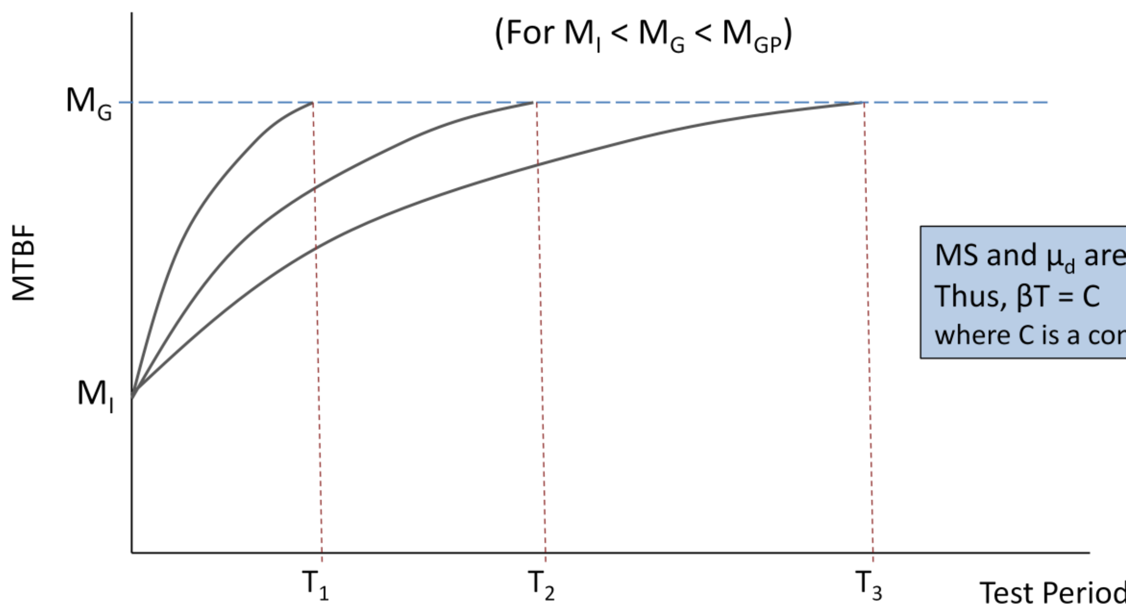

Figure 2. Property 1 Tradeoff between T and β

To address Property 1, let θ(t) be defined as follows:

$$\theta(t) = \frac{\lambda_B - h(t)}{\lambda_B} = 1 - \frac{h(t)}{h(0)} \tag{14}$$

Observe θ(t) is the expected fraction of λ_B contributed by the B-modes discovered in test by time t. Since β is a scale parameter for h(t), it is also a scale parameter for θ(t). Thus there exists a function φ(x) such that φ(βt) = θ(t) for all β > 0 and t nonnegative. Note from (14),

$$h(t) = \lambda_B\{1 - \theta(t)\} = \lambda_B\{1 - \varphi(\beta t)\} \tag{15}$$

From the definition of management strategy for $\lambda_I = (M_I)^{-1}$,

$$\lambda_B = (MS)\lambda_I = \frac{MS}{M_I} \quad , \quad \lambda_A = (1\text{-}MS)\lambda_I = \frac{1 - MS}{M_I} \tag{16}$$

From Equations (1), (15) and (16), one can express the idealized MTBF at time T, M(T), as follows:

$$M_G = M(T) = \frac{M_I}{1 - (MS)\mu_d \varphi(\beta T)} \tag{17}$$

For the scenario for Property 1, the initial and DT goal MTBF are fixed, as is the management strategy and the average FEF. Thus as T is varied, β must vary such that

$$\beta T = C \tag{18}$$

where C is a positive constant for which (17) is satisfied. A unique such constant exists since

$$M_I < M_G < M_{GP} \tag{19}$$

The existence and uniqueness of such a constant follows from the facts that (i) $\varphi(x)$ is an increasing continuous function over [0,1); (ii) $\varphi(0)=0$; and (iii) the limit of $\varphi(x)$ equals 1 as $x \rightarrow \infty$. Thus each idealized growth curve, corresponding to an h(t) from the considered class of B-mode rate of occurrence functions, can be made to yield a specified M_G for every $T > 0$, provided Equation (19) holds. Of course reducing T too far will yield an unrealistically large value of $\beta > 0$ as measured by risk metrics analogous to those considered in MIL-HDBK-189C [3] for PM2. One risk metric is based on the following fact: Under this scenario, one can show as β increases, the jump from the first MTBF benchmark step to the second MTBF step will be an increasing percentage of the total growth from M_I to M_G. Furthermore, one can show the expected number of B-modes that contribute to this jump will be decreasing as β increases.

It is interesting to consider the original MIL-HDBK-189 planning model [3] whose model equation is shown below.

$$M(T) = \left(\frac{M_1}{1 - \alpha}\right)\left(\frac{T}{t_1}\right)^{\alpha} \tag{20}$$

In this equation, M(T) denotes the DT goal MTBF to be obtained by T. Also t_1 denotes the test time in the first test phase and M_1 denotes the average MTBF over the first test phase. Just as for the h(t) scale parameter β, for a specified M(T), the smaller T is the larger the MIL-HDBK-189 growth rate α will be in Equation (20) (which does not represent the parameter α used in Section 3 for the Gamma EOS Model), indicating a larger average rate of increase in the MTBF over the test period of length T. For PM2, the relevant concern should be whether the inputted values of the five planning parameters result in a value of the scale parameter β that is realistic. Does this resulting value of β conform to one's engineering experience and understanding of the current system under development? This is comparable to asking the following with respect to the MIL-HDBK-189 Planning Model: Given current planning parameters M_1, t_1, T, and M(T) in Equation (20), is the associated value of the growth rate α

realistic? In the case of the MIL-HDBK-189 growth rate, one could compare the resulting growth rate to historically achieved growth rates to attempt to assess the risk associated with the planning parameters. For the PM2 Model, one can utilize the risk metrics in [3], as well as any available experience, to judge the plausibility of the β value implied by the input planning parameters.

5. PROPERTY 2 – TEST TIME EFFECT ON EXPECTED NUMBER OF B-MODES

The second property, corresponding to an area of concern, can be expressed as follows: As the test time T utilized to surface problem modes that contribute to growing to the DT goal MTBF increases, the PM2 model yields the same DT MTBF goal. Engineering experience indicates that as the number of corrective actions increase due to increasing T, the resultant reliability level should increase. The sensitivity analsyis conducted in the review for this property utilized the scenario in which the initial MTBF, DT goal MTBF, management strategy, and average FEF were held fixed, while T was varied. This scenario is the same as for the sponsor's first topic of concern addressed above for Property 1. Such a scenario does not pertain to the kind of engineering experience referenced by the Army organization sponsor, since it leads to a tradeoff between T and model parameter β. Changing β alters the assumed distribution of the B-mode expected initial rates of occurrence (termed the B-mode profile) arrived at by design for reliability (DFR) activities conducted prior to the growth test. This is reflected in the change of the average expected B-mode failure intensity for the B-modes surfaced over any time period $[0,t_0]$. To see this, observe by Equation (15), the average expected B-mode failure intensity for the B-modes surfaced over any time period $[0,t_0]$ is given by

$$\lambda_{B,avg}(t_0,\beta) = \frac{\lambda_B - h(t_0)}{\mu(t_0)} = \frac{\beta\varphi(\beta t_0)}{\int_0^{\beta t_0}\{1-\varphi(x)\}dx} \tag{21}$$

where μ(t) is the expected number of B-modes surfaced by t.

However, as for Property 1, it is still of interest to see whether the class of h(t) functions with a scale parameter considered in Property 1 give rise to Property 2 which was deemed counterintuitive in the review of PM2. To do so, let h(t;β) and μ(t;β) denote h(t) and μ(t), respectively, with scale parameter β.

Also let $T_2 > T_1 > 0$. Note T_i under the postulated scenario forces β_i to satisfy Equation (18) for i = 1,2. Thus $0 < \beta_2 < \beta_1$. Recall g(x) is decreasing for $x \geq 0$ and $g(\beta t) = h(t)$ for all $\beta > 0$ and $t \geq 0$. Thus, as illustrated in Figure 3,

$$\mu(T_2;\beta_2) = \int_0^{T_2} h(t;\beta_2)dt > \int_0^{T_1} h(t;\beta_2)dt > \int_0^{T_1} h(t;\beta_1)dt = \mu(T_1;\beta_1) \tag{22}$$

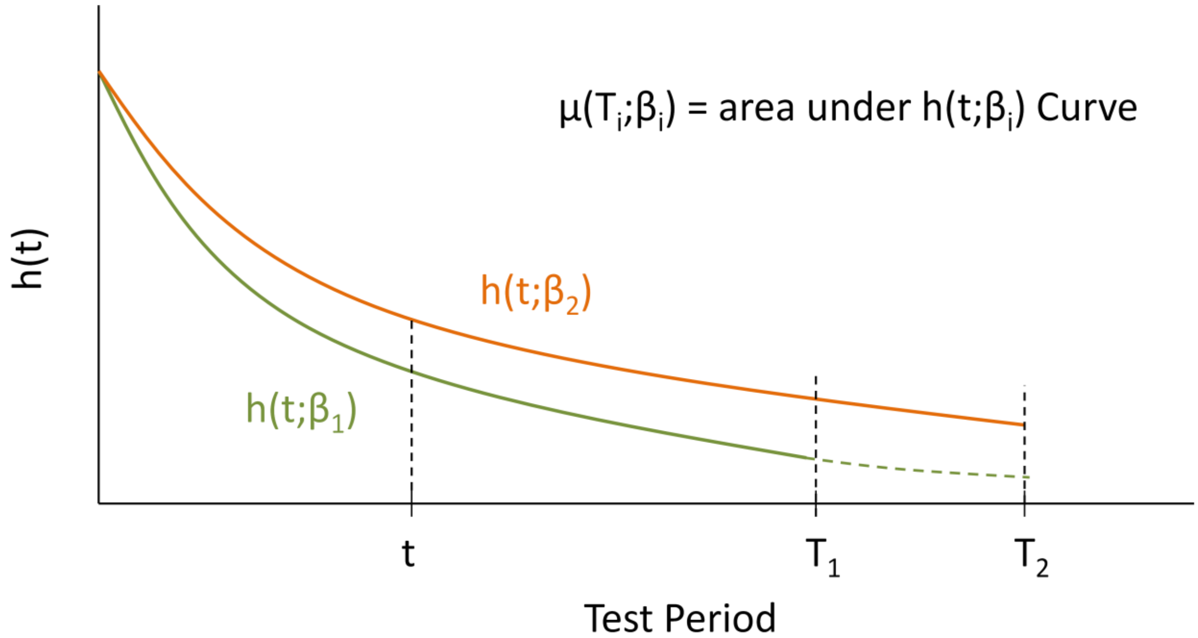

Figure 3. Expected Number of B-modes Relationship

This shows that Property 2 holds for any planning growth model for the associated scenario whose h(t) is a member of the class described in Section 3.

Note the tradeoff and resulting model behavior is not reflective of the engineering experience referenced by the Army organization since the average of the expected surfaced B-mode failure intensities is just a function of the design for reliability activities accomplished prior to the growth test and thus should remain constant across the test period. Holding β fixed along with the planning parameters except for test time T and M_G , one can inquire whether h(t) gives rise to a planning model that conforms to the referenced engineering experience. Note that for this modified scenario, the corrective actions are increasing. To see this, using Equation (15), one has:

$$\mu(T) = \lambda_B \int_0^T \{1 - \varphi(\beta t)\} dt = \frac{\lambda_B}{\beta} \int_0^{\beta T} \{1 - \varphi(x)\} dx \qquad (23)$$

Observe $\mu(T)$ is increasing via Equation (23) since λ_B and β are fixed and T is increasing. Thus the number of corrective actions increases as T increases. This, in turn, based on the referenced engineering experience, should lead to a higher achieved value of M_G. This occurs by Equation (17) since $\varphi(x)$ is an increasing function.

6. PROPERTY 3 – EFFECT OF VARYING M_I FOR A SPECIFIED GOAL MTBF

The third property addressed in the review that was an area of concern can be expressed as follows: For any time T, increasing M_I increases the expected number of corrective actions to achieve the DT goal MTBF M_G. The engineering expectation of the Army organization was that for a fixed management strategy, average FEF, and DT goal MTBF, the expected number of corrective actions to achieve M_G should decrease as the initial MTBF is raised towards M_G. This is a reasonable expectation as long as the average failure intensity associated with the B-modes surfaced over the test period T remains constant, as would be the case during the conduction of a system reliability growth test. However in the sensitivity analysis conducted in the review, M_I is varied and all the other PM2 planning inputs (including T) remain constant. Thus, by Equation (17), this scenario forces β to decrease towards zero as M_I increases towards M_G. This postulated scenario is not reflective of engineering experience associated with executing a system reliability growth test. As shown by Equation (21), the average B-mode failure intensity for B-modes surfaced during the test period from 0 to T is just a function of β and does not change over the test period. However, the postulated test scenario forces β to change, which in turn, by Equation (21), would lead to a changing average B-mode failure intensity for the test period. This postulated scenario in which a forced tradeoff between M_I and β occurs will be explored in Section 6.2 for the class of h(t) functions discussed in Section 3 that are used in Equation (1) to generate the idealized failure intensity and associated MTBF step reliability growth planning curve.

6.1 Scenario Pertinent to a Reliability Growth Test as M_I Varies. First, consider an altered scenario that corresponds to engineering experience associated with conducting a reliability growth test over a test period T. In this scenario the management strategy, average FEF, and DT goal MTBF M_G shall be held constant as the initial MTBF is raised towards M_G. To have the scenario be pertinent to engineering experience associated with conducting a reliability growth test, the average expected failure intensity of B-modes surfaced over any test interval is fixed and determined by the design for reliability activities prior to the start of the test. Thus β is fixed. However, by Equation (17), the product βT must be decreasing as M_I is increased towards M_G. Thus the amount of test time T required to obtain the DT goal MTBF is decreasing, in accordance with engineering experience. Also, since T and $\lambda_B = (MS)(\lambda_I)$ are decreasing and β is fixed , the expected number of B-modes surfaced by T is decreasing by Equation (23). This in turn implies that the expected number of corrective actions required to obtain the DT goal MTBF should be decreasing as the initial MTBF is raised, which conforms to engineering experience.

6.2 Effect on Expected Number of B-modes as M_I/M_G Varies. Now consider the scenario associated with Property 3, i.e., all planning parameters are assumed fixed except for M_I. This section assumes that $0 < (MS)\mu_d < 1$. Contrary to what Property 3 states in the review, for this scenario one can show that there exists η_1 and η_2 with $1-(MS)\mu_d < \eta_1 \leq \eta_2 < 1$ that satisfy the following: As M_I is increased towards M_G, the expected number of B-modes by T is an increasing function of $\eta = \frac{M_I}{M_G}$ for $1-(MS)\mu_d < \eta \leq \eta_1$ and, when $(MS)\mu_d > 0.5$, a decreasing function of η for $\eta_2 \leq \eta < 1$.

In the following, let $\eta = f(y)$ denote the function that relates η to $y = \beta T$, where $\beta > 0$ is the scale parameter for h(t) and T is fixed. By Equation (17) in Section 4 one has

$$\frac{M_I}{M_G} = \eta = f(y) = 1 - (MS)\mu_d\varphi(y) \tag{24}$$

Since $\varphi(y)$ is an increasing function of $y > 0$, η is a decreasing function of $y > 0$. By Equation (24),

$$f(0) = 1 \text{ and } \lim_{y\to\infty} f(y) = 1 - (MS)\mu_d \tag{25}$$

where $0 < 1\text{-}(MS)\mu_d < 1$.

Also note from Equation (17) and the definition of y, one has for $y > 0$,

$$\lambda_B = (MS)\lambda_I = (MS)\left\{\frac{\rho_G}{1 - (MS)\mu_d\varphi(y)}\right\} \text{ and } \beta = yT^{-1} \tag{26}$$

For this section, let $\mu(T;y)$ denote the expected number of B-modes surfaced by T where λ_B and β are given by the equations in (26). Next, it will be shown that there exists y_1 and y_2 such that $0 < y_2 \le y_1 < \infty$ for which $\mu(T; y)$ is an increasing function of y for $0 < y \le y_2$, provided $(MS)\mu_d > 0.5$, and a decreasing function of y for $y_1 \le y$. Since $\eta = f(y)$ and f(y) is a decreasing function for $y > 0$, it will follow that $\mu(T)$ is an increasing function of η for $1\text{-}(MS)\mu_d < \eta \le \eta_1$ and a decreasing function of η for $\eta_2 \le \eta < 1$. These values of η are given by $\eta_i = f(y_i)$ for i =1,2 (see Figure 4).

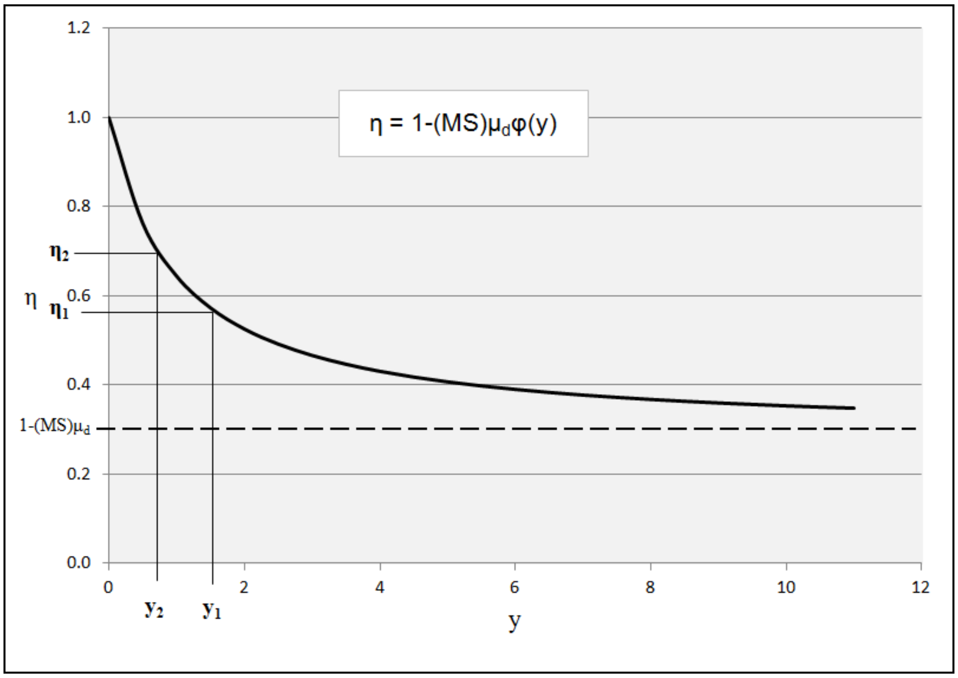

Figure 4. Relationship Between $\eta = M_I/M_G$ and $y = \beta T$

6.2.1 Effect on Expected Number of B-modes as M_I/M_G Approaches One. It will now be shown that when $(MS)\mu_d > 0.5$ there exists a $y_2 > 0$ such that $\mu(T;y)$ is an increasing function of y for $0 < y \leq y_2$. By Equations (23) and (26) one has

$$\mu(T;y) = \left\{\frac{(MS)\rho_G T}{1-(MS)\mu_d \varphi(y)}\right\}\left\{1-\frac{1}{y}\int_0^y \varphi(x)dx\right\} \tag{27}$$

Using Equation (27), one can obtain the derivative of $\mu(T;y)$ with respect to y. It can be shown that

$$\frac{d\mu(T;y)}{dy} > 0 \leftrightarrow Q(y)\left\{1-\frac{1}{y}\int_0^y \varphi(x)dx\right\} > \frac{1}{y}\left\{\varphi(y)-\frac{1}{y}\int_0^y \varphi(x)dx\right\} \tag{28}$$

where

$$Q(y) = \frac{(MS)\mu_d \frac{d\varphi(y)}{dy}}{1-(MS)\mu_d \varphi(y)} \tag{29}$$

Note the limit of Q(y) as y goes to zero equals $(MS)\mu_d L$ where

$$L = \lim_{y\to 0}\frac{d\varphi(y)}{dy} \tag{30}$$

The limit in Equation (30) exists and is a finite positive number since h(t) satisfies the fundamental properties (i) through (iii).

Next consider the limit as $y \to 0$ of the second factor on the left-hand side of the second inequality in (28). This can be shown to equal to 1 by applying l'Hôpital's Rule. Applying l'Hôpital's Rule as $y \to 0$ to the right-hand side of the second inequality in (28) yields (0.5)L. Thus as y goes to zero, the limit of the left-hand side of the second inequality in (28) is greater than the corresponding limit of the right-hand side of the second inequality in (28), provided $(MS)\mu_d > 0.5$.

Therefore, for $(MS)\mu_d > 0.5$, there exists y_2 such that

$$\frac{d\mu(T;y)}{dy} > 0 \tag{31}$$

for $0 < y < y_2$. This implies, as discussed above, $\mu(T)$ is a decreasing function of $\frac{M_I}{M_G} = \eta = f(y)$ for $\eta_2 \leq \eta < 1$, where $\eta_2 = f(y_2)$. This is in contradiction to the corresponding statement of Property 3 in the review of PM2.

6.2.2 Effect on Expected Number of B-modes as M_I/M_G Increases Near 1-$(MS)\mu_d$. Next, it will be shown there exists $y_1 \geq y_2$ such that $\mu(T; y)$ is decreasing for $y \geq y_1$. In the following let $y > 0$. By (28) one has

$$\frac{d\mu(T;y)}{dy} < 0 \leftrightarrow Q(y)\left\{1 - \frac{1}{y}\int_0^y \varphi(x)dx\right\} < \frac{1}{y}\left\{\varphi(y) - \frac{1}{y}\int_0^y \varphi(x)dx\right\} \tag{32}$$

Note,

$$1 - \frac{1}{y}\int_0^y \varphi(x)dx = \frac{1}{y}\left\{y - \int_0^y \varphi(x)dx\right\} > 0 \tag{33}$$

Thus by Equations (32) and (33) one has

$$\frac{d\mu(T;y)}{dy} < 0 \leftrightarrow Q(y) < \frac{\varphi(y) - \frac{1}{y}\int_0^y \varphi(x)dx}{y - \int_0^y \varphi(x)dx} \tag{34}$$

The second inequality in (34) can be written as follows:

$$yQ(y) < \frac{y\varphi(y) - \int_0^y \varphi(x)dx}{y - \int_0^y \varphi(x)dx} \tag{35}$$

Inequality (35) is equivalent to the inequality below:

$$1 - \frac{y\varphi(y) - \int_0^y \varphi(x)dx}{y - \int_0^y \varphi(x)dx} < 1 - yQ(y) \tag{36}$$

Note

$$1 - \frac{y\varphi(y) - \int_0^y \varphi(x)dx}{y - \int_0^y \varphi(x)dx} = \frac{y - \int_0^y \varphi(x)dx - y\varphi(y) + \int_0^y \varphi(x)dx}{y - \int_0^y \varphi(x)dx} = \frac{y\{1 - \varphi(y)\}}{\int_0^y \{1 - \varphi(x)dx\}} \tag{37}$$

From (37) and the equivalence of the inequalities in (34), (35) and (36) one obtains the following for y > 0:

$$\frac{d\mu(T;y)}{dy} < 0 \leftrightarrow \frac{y\{1 - \varphi(y)\}}{\int_0^y \{1 - \varphi(x)dx\}} < 1 - yQ(y) \tag{38}$$

By (29),

$$yQ(y) = \frac{(MS)\mu_d y \frac{d\varphi(y)}{dy}}{1 - (MS)\mu_d \varphi(y)} \tag{39}$$

Note φ(y) is a nonnegative, increasing, bounded, and twice differentiable concave function. Thus by Proposition A.1 in Appendix A one has

$$\lim_{y\to\infty} y\frac{d\varphi(y)}{dy} = 0 \tag{40}$$

Also, $\lim_{y\to\infty}\varphi(y) = 1$. Thus the right hand side in the second inequality in (38), by (39) and (40), satisfies

$$\lim_{y\to\infty}\{1 - yQ(y)\} = 1 \tag{41}$$

To consider the limit of $\frac{y\{1-\varphi(y)\}}{\int_0^y\{1-\varphi(x)dx\}}$ as y→∞ in the second inequality in (38), some notation will first be established.

Let z ≥ 0. It will be convenient to express h(z) by h(z; p_1, p_2) where p_1 and p_2 are positive parameters that are held fixed as z → ∞ . Also, it is assumed that any additional minimal set of parameters that are required to define h are kept fix. With regard to p_1 and p_2, this notation will imply that p_1 is a scale parameter for h(z; p_1, p_2) and p_2 = h(0; p_1, p_2) . Also define θ(z; p_1 , p_2) = {h(0; p_1, p_2) – h(z; p_1, p_2)} ÷ h(0; p_1, p_2). Note θ(0; p_1, p_2) = 0. Also, by the assumptions imposed on h and the definition of θ, it follows that the values of θ do not depend on the value of p_2. Thus, the value θ(z; p_1, p_2) will simply be expressed as θ(z; p_1). One can show that p_1 is a scale parameter for θ(z; p_1) since p_1 is a scale parameter for h(z; p_1, p_2). More precisely, it can shown there exists a function φ(x) such that φ(p_1z) = θ(z; p_1) for all z ≥ 0 and p_1 > 0 , for each fixed minimal set of parameters that include p_1, p_2 and define h.

Recall by the definition of y in Section (6.2), one has y = βT. Thus

$$\frac{y\{1-\varphi(y)\}}{\int_0^y\{1-\varphi(x)dx\}} = \frac{\beta T\{1-\varphi(\beta T)\}}{\int_0^{\beta T}\{1-\varphi(x)dx\}} = \frac{\beta\{1-\varphi(T\beta)\}}{(\frac{1}{T})\int_0^{T\beta}\{1-\varphi(x)dx\}} \tag{42}$$

Observe, by (15),

$$1-\varphi(T\beta) = 1-\theta(\beta;T) = h(\beta;T,1) \tag{43}$$

Let z = x ÷ T. Then

$$\int_0^{T\beta}\{1-\varphi(x)dx\} = T\int_0^{\beta}\{1-\varphi(Tz)\}dz = T\int_0^{\beta}\{1-\theta(z;T)\}dz \tag{44}$$

Thus

$$\left(\frac{1}{T}\right)\int_0^{T\beta}\{1-\varphi(x)dx\} = \left(\frac{1}{T}\right)\left[T\int_0^{\beta}\{1-\theta(z;T)\}dz\right] \tag{45}$$

Equations (15), (42), (43), and (45) yield

$$\frac{y\{1-\varphi(y)\}}{\int_0^y \{1-\varphi(x)dx\}} = \frac{\beta h(\beta;T,1)}{\int_0^\beta \{1-\theta(z;T)\}dz} = \frac{\beta h(\beta;T,1)}{\int_0^\beta h(z;T,1)dz} = \frac{h(\beta;T,1)}{\frac{\mu(\beta;T,1)}{\beta}} \tag{46}$$

Equation (46), the second equation in (26) and the previously stated assumption that $\lim_{t\to\infty}[\, h(t) \div \{\, \mu(t)/t\}] < 1$
when the parameters defining h are fixed as t→∞ imply the following:

$$\lim_{y\to\infty} \frac{y\{1-\varphi(y)\}}{\int_0^y \{1-\varphi(x)dx\}} = \lim_{\beta\to\infty} \frac{h(\beta;T,1)}{\frac{\mu(\beta;T,1)}{\beta}} < 1 \tag{47}$$

Thus Equations (38), (41), and (47) demonstrate there exists $y_1 \geq y_2$ such that μ(T; y) is decreasing for $y \geq y_1$. As discussed earlier, this implies that μ(T) is an increasing function of $\frac{M_I}{M_G} = \eta = f(y) = 1 - (MS)\mu_d \varphi(y)$ for $1-(MS)\mu_d < \eta \leq \eta_1$, where $\eta_1 = f(y_1)$.

7. PROPERTY 4 – EFFECT OF LIMITING THE RATIO OF M_G TO M_{GP}

The last of the four "counterintuitive" properties in the review can be stated as follows: Using the criterion that an acceptable growth plan should have the DT goal MTBF no higher than 80% of the growth potential MTBF, the PM2 Model limits the use of commercial-off-the-shelf items (COTS) from a low of 0% to a maximum of 70% of the initial system failure rate. In the review the low was actually mistakenly stated to be 38% which is the percent of the initial MTBF to the goal MTBF that limits the use of COTS to approximately zero percent of the initial system failure rate. Also, the DT goal MTBF was taken to be the Initial Operational Test (IOT) MTBF goal. The IOT is a demonstration test conducted by service personnel in simulated mission scenarios after the conclusion of the DT test program (see Figure 1). This IOT goal needs to be set sufficiently above the requirement MTBF to be demonstrated during the IOT, M_R, to have a reasonable chance of being able to successfully demonstrate the requirement with statistical confidence during the IOT. Additionally, the DT MTBF goal is set higher than the IOT MTBF goal, $M_{G,IOT}$, if a drop γ in reliability from the DT to the IOT environment is expected due to more realistic testing, i.e., $M_{G,IOT} = (1-\gamma)M_G$. Property 4 in the review was stated for $\gamma = 0$. The PM2 software characterizes as high risk a ratio of the DT goal to the growth potential greater than 0.80. However this is general guidance, independent of the particular growth model. The fact that the COTS percent contribution to the initial system failure rate must be less than 71% (not 70% as stated in the review) under the 0.80 guidance is independent of the PM2 model. It is shown below this guidance results in the same 71% COTS restriction for a wide class of growth planning models.

The allowable COTS contribution also depends on the assumed drop γ and value of μ_d. Although not stated, it appears that the review set $\mu_d = 0.70$, and $\gamma = 0.0$, and assumed the failure modes associated with COTS items were all A-modes and that these modes were the only A-modes. In general, neither of these assumptions need be true. Under these assumptions, the review results were closely reproduced by use of Equation (49) below. For this case, the portion of the initial system failure rate due to COTS is $\frac{\lambda_A}{\lambda_I}$ = (1-MS). This ratio can be calculated as a function of $\frac{M_I}{M_{G,IOT}}$, by solving Equation (48) for MS:

$$\frac{M_G}{M_{GP}} = \frac{M_G}{\left\{\dfrac{M_I}{1-(MS)\mu_d}\right\}} = \frac{1-(MS)\mu_d}{M_I} M_G \tag{48}$$

Utilizing the expression for MS, one obtains

$$\frac{\lambda_A}{\lambda_I} = 1 - \left(\frac{1}{\mu_d}\right)\left[1 - \left\{(1-\gamma)\left(\frac{M_G}{M_{GP}}\right)\right\}\left(\frac{M_I}{M_{G,IOT}}\right)\right] \tag{49}$$

The only assumptions about the reliability growth model used to obtain Equation (49) are the following: (a) The idealized failure intensity associated with the reliability growth model is given by Equation (1); and (b) the B-mode rate of occurrence function in Equation (1) is a positive decreasing function of $t \geq 0$ with a finite initial value $\lambda_B = h(0)$. Also $h(t) \to 0$ as $t \to \infty$.

The review did not state the equation utilized to calculate $\frac{\lambda_A}{\lambda_I}$ as a function of $\frac{M_I}{M_{G,IOT}}$. From the above, it is apparent that the "counterintuitive" property concerning COTS ascribed to PM2 is shared by a large class of growth models whose h(t) function satisfies the basic properties described in (b) above.

8. REMARKS

The four properties of PM2-Continuous identified in a review as being counterintuitive were addressed in the paper. It was demonstrated that these properties are shared by reliability growth planning models that are based on a large class of B-mode rate of occurrence functions h(t) that were advocated by Miller [5] for use in modeling the occurrence of new software bugs in reliability growth testing. The label of "counterintuitive" is based on a misapplication of the engineering experience associated with conducting a reliability growth test. In such a test the B-mode initial rates of occurrence arrived at by up-front design for reliability activities are fixed. However, the implied model parameter trade-offs associated with these "counterintuitive" properties change the implied B-mode profile as reflected in a change to the average of the expected surfaced B-mode failure intensities. The paper demonstrated that if the scenarios associated with these properties are modified in such a way that this average remains constant, then a planning model generated by any h(t) in the class described conforms to the engineering experience referenced by the review's sponsor.

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APPENDIX A – LIMIT PROPOSITION FOR BOUNDED CONCAVE FUNCTIONS

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LIMIT PROPOSITION FOR BOUNDED CONCAVE FUNCTIONS

Proposition A.1. A nonnegative, increasing, bounded, and twice differentiable concave function $\psi(y)$ defined for $y \geq 0$ satisfies the following: $\lim_{y \to \infty} y\,\psi'(y) = 0$.

Proof. Let $y \geq 0$. To show the above we shall utilize a function w(y) whose slope is $y\psi'(y)$. Thus it will be shown that $\lim_{y \to \infty} w'(y) = 0$. One can choose w(y) as follows:

$$w(y) = \int_0^y x\psi'(x)dx \tag{A.1}$$

Note $w'(y) = y\psi'(y)$. A useful representation for w(y) can be obtained by integration by parts.

Observe $\int x\psi'(x)dx = \int udv = uv - \int vdu$ for $u = x$ and $dv = \psi'(x)dx$. Thus $du = dx$ and one can choose $v = \psi(x)$. Also x = 0 implies u = 0 and x = y implies u = y. From the above one obtains the following:

$$\int_0^y x\psi'(x)dx = x\psi(x)\Big|_0^y - \int_0^y \psi(x)dx = y\psi(y) - \int_0^y \psi(x)dx \tag{A.2}$$

Since $0 \leq \psi(y)$ is increasing and bounded, $\psi_\infty = \lim_{y \to \infty} \psi(y)$ exists and is a finite positive number. From the concavity of $\psi(x)$ it follows that $\psi'(x)$ is decreasing. Since $\psi'(x)$ is decreasing for $x > 0$ one has the following for $y > 0$ (see Figure 5):

$$0 < y\psi'(y) < \int_0^y \psi'(x)dx = \psi(y) < \psi_\infty < \infty \tag{A.3}$$

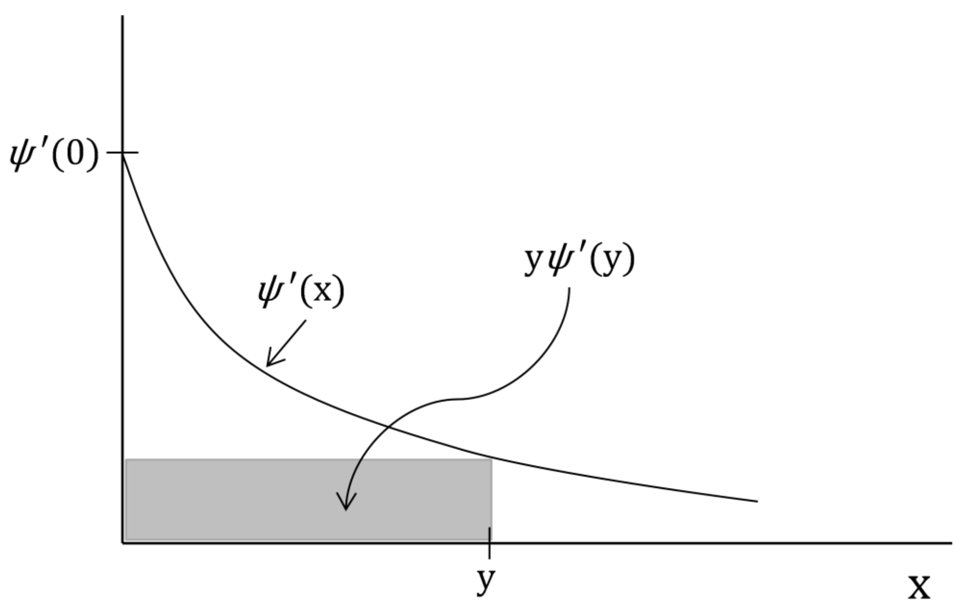

Figure 5. Graphical Depiction of (A.3)

Next it will be shown that $\lim_{y\to\infty} y\psi'(y)$ exists. Observe

$$\psi(y) = \int_0^y \psi'(x)dx = \int_0^y \{\psi'(x) - \psi'(y)\}dx + y\psi'(y) \tag{A.4}$$

Thus

$$\int_0^y \{\psi'(x) - \psi'(y)\}dx = \psi(y) - y\psi'(y) \tag{A.5}$$

The inequalities in (A.3) show that ψ(y) is bounded above by ψ_∞ and that $y\psi'(y)$ lies in the finite open interval (0, ψ_∞). Thus (A.5) demonstrates that $\int_0^y \{\psi'(x) - \psi'(y)\}dx$ is bounded above. Also this integral is positive for $y > 0$ since $\psi'(x) > \psi'(y)$ for $0 \le x < y$ (as displayed in Figure 5). From Figure 5, one can see that this integral is an increasing function of $y > 0$. This can also be demonstrated using (A.5). By (A.5) one obtains the following:

$$\frac{d}{dy}\int_0^y \{\psi'(x) - \psi'(y)\}\,dx = \psi'(y) - \{\psi'(y) + y\psi''(y)\} = -y\psi''(y) > 0$$

Since $\int_0^y \{\psi'(x) - \psi'(y)\}dx$ has been shown to be a positive increasing bounded function for $y > 0$, it follows that $\lim_{y\to\infty} \int_0^y \{\psi'(x) - \psi'(y)\}dx$ exists and is finite. Note from (A.4) one obtains

$$y\psi'(y) = \psi(y) - \int_0^y \{\psi'(x) - \psi'(y)\}dx) \tag{A.6}$$

Since it has been demonstrated that ψ(y) and $\int_0^y \{\psi'(x) - \psi'(y)\}dx)$ have finite limits as $y \to \infty$, it follows from (A.6) that $\lim_{y\to\infty} y\psi'(y)$ exists. Also from (A.3) one has

$$0 \le \lim_{y\to\infty} y\psi'(y) \le \psi_\infty < \infty$$

Let $s = \lim_{x\to\infty} x\psi'(x)$. Suppose $s > 0$. Then there exists $y_1 > 0$ such that

$$0 < (0.5)s < x\psi'(x) \text{for } x \ge y_1 \tag{A.7}$$

Let $y > y_1$. Then

$$\int_0^y x\psi'(x)dx = \int_0^{y_1} x\psi'(x)dx + \int_{y_1}^y x\psi'(x)dx > \int_0^{y_1} x\psi'(x)dx + \int_{y_1}^y (0.5)s\,dx \tag{A.8}$$

Thus, by (A.8),

$$\int_0^y x\psi'(x)dx > \int_0^{y_1} x\psi'(x)dx + (0.5)s(y - y_1) \tag{A.9}$$

Therefore, by (A.9), for $y > y_1$ one obtains

$$y^{-1}\int_0^y x\psi'(x)dx > y^{-1}\int_0^{y_1} x\psi'(x)dx + (0.5)s\left(1-\frac{y_1}{y}\right) \quad \text{(A.10)}$$

Note

$$\lim_{y\to\infty}\left\{y^{-1}\int_0^{y_1} x\psi'(x)dx + (0.5)s\left(1-\frac{y_1}{y}\right)\right\} = (0.5)s \quad \text{(A.11)}$$

Thus, by (A.10) and (A.11), there exists $y_2 \geq y_1$ such that for $y > y_2$,

$$y^{-1}\int_0^y x\psi'(x)dx > (0.4)s > 0 \quad \text{(A.12)}$$

By (A.2) one has

$$y^{-1}\int_0^y x\psi'(x)dx = \psi(y) - y^{-1}\int_0^y \psi(x)dx \quad \text{(A.13)}$$

Note $\lim_{y\to\infty}\psi(y) = \psi_\infty \epsilon\ (0,\infty)$. Therefore $\lim_{y\to\infty}\int_0^y \psi(x)dx = \infty$. Thus by l'Hôpital's Rule,

$$\lim_{y\to\infty}\frac{\int_0^y \psi(x)dx}{y} = \lim_{y\to\infty}\frac{\frac{d}{dy}\int_0^y \psi(x)dx}{\frac{d}{dy}y} = \lim_{y\to\infty}\psi(y) = \psi_\infty \quad \text{(A.14)}$$

It follows from (A.13) and (A.14) that

$$\lim_{y\to\infty} y^{-1}\int_0^y x\psi'(x)dx = \lim_{y\to\infty}\psi(y) - \lim_{y\to\infty}\left\{\frac{\int_0^y \psi(x)dx}{y}\right\} = \psi_\infty - \psi_\infty = 0 \quad \text{(A.15)}$$

Observe (A.15) contradicts (A.12). Thus $\lim_{y\to\infty} y\,\psi'(y) = s = 0$.

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